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Non-analyticity in scale in the planar limit of QCD

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Using methods of numerical Lattice Gauge Theory we show that in the limit of a large number of colors, properly regularized Wilson loops have an eigenvalue distribution which changes non-analytically as the overall size of the loop is increased. This establishes a large-N phase transition in continuum planar gauge theory, a fact whose precise implications remain to be worked out.

Intuitively, one expects parallel transport round a closed curve in four-dimensional SU(N) pure gauge theory with $\theta_{CP} = 0$ to be close to identity for small curves and far from identity for large curves. In this letter we make this idea concrete and find that small and large loops are separated by a large-N phase transition. The possibility of a new type of non-analyticity entering SU(N) gauge theory in the 't Hooft limit [1] $N \to \infty$ has preoccupied researchers for a long time; here we present a class of examples where this phenomenon occurs. The numerical evidence is sufficiently convincing to view the effect as an exact property of the large-N limit. We start presenting a simple lattice result and let it lead us to the above conclusion. After that, we put the result in historical context, mentioning some of the main papers this work is related to.

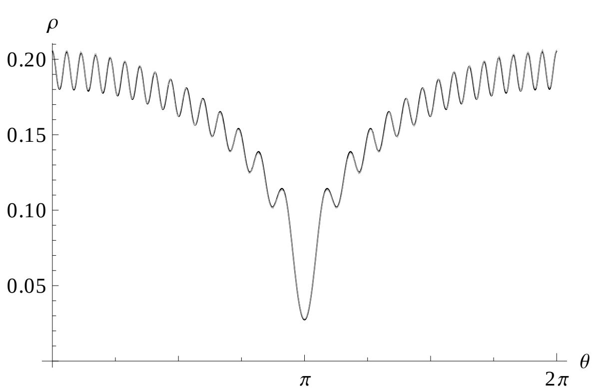

FIG. 1. Histogram of eigenvalue angles for a Wilson loop.

The gray line in Fig. 1 is the histogram for the angles of the eigenvalues of a square loop in SU(29) lattice gauge theory with side length equivalent to about 0.54 fermi in QCD units and viewed at a resolution (thickness) of about 0.15 fermi. The black line is the eigenvalue-angle density determined by the heat-kernel function (HK) for SU(29) which depends on a single parameter t, to be defined later (in the plot, $t \approx 4$). The two curves cannot be distinguished in the figure, so the HK approximates the data well, after adjusting t. The data is sufficiently accurate to check whether the HK might provide an exact description. This is definitely ruled out by a χ^2 analysis.

Some definitions are in order now. The Wilson loop matrix associated with a closed spacetime curve $\mathcal{C}$ is

$$W_r(\mathcal{C}, x, s) = \mathcal{P} \exp \left(i \oint_{x;\mathcal{C}}^{x} A_\mu^r(y, s) dy_\mu \right) \in \mathrm{SU}(N),$$

where r denotes an SU(N) representation, x is a point on $\mathcal{C}$, and $s > 0$ denotes a "smearing parameter" of dimension length squared ($\sqrt{s}$ determines the thickness of the loop). Smearing is required to make all $W_r(\mathcal{C}, x, s)$ finite SU(N) matrices with operator-valued entries. The gray line on the previous plot shows

$$\rho_N(\theta; \mathcal{C}, s) = \frac{1}{2\pi N} \sum_{i=1}^{N} \langle \delta(\theta - \theta_i(\mathcal{C}, s)) \rangle .$$

The $\theta_i(\mathcal{C}, s)$ are angles locating the eigenvalues of $W_f(\mathcal{C}, x, s)$ (f denotes the fundamental representation) on the unit circle; they do not depend on the choice of x. After averaging, also the dependence on the location and the orientation of the loop drops out. With the CP violating θ_{CP} parameter set to 0, ρ_N is invariant under $\theta \to 2\pi - \theta$.

Smearing is defined as follows: One starts with five-dimensional gauge fields on $\mathbb{R}^4 \times \mathbb{R}_+$; the smearing parameter s lives on the $\mathbb{R}_+$. The usual quantum fields are denoted by $B_\mu^f(x)$ and reside on the $\mathbb{R}^4$-boundary. The $A_\mu^f(x, s)$ are defined for $s \geq 0$ by

$$F_{\mu,s}^f = D_\nu^{\mathrm{adjoint}} F_{\mu,\nu}^f \quad \text{with} \quad A_\mu^f(x, s = 0) = B_\mu^f(x).$$

The 5D gauge freedom is reduced to a 4D one by $A_s^f(x, s) = 0$. At $s > 0$, all divergences coming from coinciding spacetime points in products of renormalized elementary fields are eliminated by a limitation on the resolution of the observer, parametrized by s. Renormalization of the boundary quantum-fields $B_\mu^f(x)$ proceeds as usual. The definition of smearing easily extends to any finite UV cutoff including the lattice: replace $D_\nu F_{\mu,\nu}$ by the variation of the action with the UV cutoff in place. Smearing extends formally to loop space, with $\mathcal{C}$ parametrized by σ,

$$\partial_s \mathrm{Tr}\langle W_f(\mathcal{C}, s)\rangle = \oint_\sigma \frac{\delta^2 \, \mathrm{Tr}\langle W_f(\mathcal{C}, s)\rangle}{\delta x_\mu^2(\sigma)} \equiv \hat{L} \, \mathrm{Tr}\langle W_f(\mathcal{C}, s)\rangle .$$

$\hat{L}$ is the Lévy Loop Laplacian appearing in the Makeenko-Migdal (MM) equations. Were a string representation

of $\hat{L}$ found, diffusion in loop space would become well defined and field theory and string theory could refer then to the same non-singular object.

The HK (heat kernel) probability density (w.r.t. the Haar measure) for an SU(N) matrix W is

$$\mathcal{P}_N^{\rm HK}(W,t) = \sum_{\text{all irred. } r} d_r \chi_r(W) e^{-\frac{t}{N}C_2(r)},$$

implying $\langle \chi_r(W)\rangle = d_r e^{-\frac{t}{N}C_2(r)}$. The parameter t is a "diffusion time" and d_r, $C_2(r)$ are the dimension and the quadratic Casimir of the irreducible representation r (in Fig. 1, $t = 3.881$). The heat kernel represents a multiplicative random walk on the SU(N) group manifold emanating from the identity. The HK single eigenvalue distribution $2\pi\rho_N^{\rm HK}(\theta,t)$ is given by

$$1+\frac{2}{N}\sum_{p=0}^{N-1}(-1)^p\sum_{q=0}^{\infty}\cos((p+q+1)\theta)d(p,q)e^{-\frac{t}{N}C(p,q)},$$

where $C(p,q) = \frac{1}{2}(p+q+1)\left(N - \frac{p+q+1}{N} + q - p\right)$ and $d(p,q) = \frac{(N+q)!}{p!q!(N-p-1)!}\frac{1}{p+q+1}$.

The HK represents the data very well, but is not exact. One could explain this by postulating "Casimir dominance": $\mathrm{Tr}\,\langle W_r(\mathcal{C},s)\rangle \approx d_r\ e^{-C_2(r)\mathfrak{S}(\mathcal{C},s)}$, with a r-independent $\mathfrak{S}(\mathcal{C},s)$. This approximation must break down for very large loops, where screening effects come in and the loop is dominated by the area term. Then, only the N-ality of r should matter. However, both in perturbation theory and at intermediate scales (up to 2 fermi), Casimir dominance is known to be a good approximation. All our data is in the range (0,1) fermi because the large-N transition is roughly in the middle of this segment.

We proceed to make the case that this indication of approximate Casimir dominance can be replaced by an exact statement in the large-N limit, namely that the HK formula is exact in a precise "large-N universality" sense to be explained below. The main point is that the shape of the eigenvalue-angle distribution of smeared Wilson matrices associated with uncomplicated loops is governed by two opposite tendencies: random-matrix eigenvalue repulsion and asymptotic-freedom attraction to unity. This produces N alternating peaks and valleys with a swing of order $\frac{1}{N}$ between them. So, most of the structure one sees in Fig. 1 is determined by a generic mechanism. Rather than the oscillations, the truly interesting feature is the deep well around π. For a smaller loop, this well would be deeper and wider. Taking N to infinity eliminates the more obvious details and leaves the essential features: for a small loop the valley around π flattens out at $N=\infty$ and the eigenvalue density is zero there. For a large loop, the eigenvalue density is non zero around the entire unit circle. In the HK case, the role of overall loop size is played by t and the transition between a gap-less and a gapped spectrum occurs at $t=4$. Large-N universality will be a statement about the large-N behavior of the gauge theory data in the neighborhood of a loop size that would be critical at $N=\infty$.

To concentrate on the region of interest we need to choose an observable that is particularly sensitive to eigenvalues close to -1:

$$\mathcal{O}_N(y,\mathcal{C},s) = \left\langle \det\left(e^{\frac{y}{2}} + e^{-\frac{y}{2}}W_f(\mathcal{C},s)\right)\right\rangle.$$

This observable generates all Wilson loop expectation values in the k-antisymmetric irred. representations of SU(N) and is given by $\sum_{k=0}^{N} e^{\left(\frac{N}{2}-k\right)y}\langle\chi_k^{\rm asym}(W_f(\mathcal{C},s))\rangle$.

At the transition point, $\log\mathcal{O}_N$ will have a non-analyticity at $y=0$ when $N=\infty$. Therefore, we consider the expansion

$$\mathcal{O}_N(y,\mathcal{C},s) = a_0(\mathcal{C},s) + a_1(\mathcal{C},s)y^2 + a_2(\mathcal{C},s)y^4 + O\left(y^6\right)$$

and use $\omega_N(\mathcal{C},s) = \frac{a_0(\mathcal{C},s)a_2(\mathcal{C},s)}{a_1^2(\mathcal{C},s)}$ as our new observable. In the HK case, with $\tau \equiv t(1+1/N)$, $\phi_N^{\rm HK}(y,\tau) = -\frac{1}{N}\left(\frac{\partial}{\partial y}\log(\mathcal{O}_N^{\rm HK}(y,\tau)\right)$ obeys Burgers' equation:

$$\partial_\tau\phi_N^{\rm HK} + \phi_N^{\rm HK}\partial_y\phi_N^{\rm HK} = \frac{1}{2N}\partial_y^2\phi_N^{\rm HK}.$$

N enters only via the viscosity, which is $\frac{1}{2N}$. We may replace the term "large-N universality" by "Burgers universality". Our observable in the HK case is plotted in Fig. 2 for different values of N.

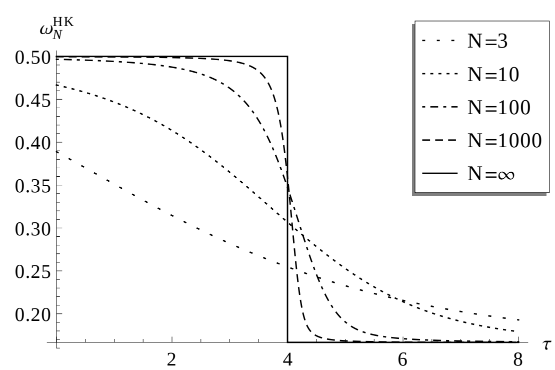

FIG. 2. Development of a jump singularity in the HK case.

Figure 2 shows that $\omega_N^{\rm HK}(\tau)$ becomes a θ-function at infinite N but the singular behavior develops slowly with N. Burgers' equation implies that the jump in the θ-function causes $\partial_y\phi_\infty^{\rm HK}(y=0,\tau)$, which is finite and negative for $\tau<4$, to blow up when τ reaches $\tau=4$. We expect a similar behavior in the four-dimensional gauge theory. To be specific, Burgers universality means, e.g., that the following predictions about the large-N limit hold exactly in the four-dimensional gauge theory, where criticality sets in at an overall loop size

$l = l_c$: $\lim_{N\to\infty} N^{-\frac{3}{2}} \left.\frac{a_1}{a_0}\right|_{l=l_c} = \frac{1}{8}\sqrt{\frac{3}{2}}\frac{1}{K}$, (here, $K \equiv \frac{1}{4\pi}\Gamma^2\left(\frac{1}{4}\right) \approx 1.046$); $\lim_{N\to\infty} N^{-\frac{3}{2}} \left.\frac{a_2}{a_1}\right|_{l=l_c} = \frac{1}{24}\sqrt{\frac{3}{2}}K$; $\lim_{N\to\infty} \omega_N|_{l=l_c} = \frac{1}{3}K^2$. Here, the variation w.r.t. the overall size l is taken at constant loop shape. The roots of $\mathcal{O}_N(y)$ are all on the imaginary axis (as a consequence of the Lee-Yang theorem). Another universal property is that in the critical regime (around $y = 0$, $l = l_c$) these roots scale like $N^{-\frac{3}{4}}$.

Our objective was to establish numerically that the transition occurs and that the universal predictions hold in the continuum limit of a lattice gauge theory with standard single plaquette Wilson action. The lattice coupling, traditionally denoted by β, is determined by the inverse 't Hooft coupling $b = \frac{\beta}{2N^2}$. As N varies, simulations are carried out in varying ranges of b, all contained in the segment $[0.348, 0.380]$ with upper limits determined by the lattice volumes, V. We have carried out simulations at $N = 11, 19, 29$ on hypercubic lattice volumes, V, $(12^4, 14^4, 18^4)$, $(10^4, 12^4, 13^4, 14^4)$, $(8^4, 10^4, 12^4)$, respectively. The volume size determined the maximal b allowed in order to stay in the confined phase at infinite N. For finite and fixed N, at b values close to the maximal allowed one, sizable finite-volume effects became evident; they were consistent with an exponential dependence on the linear size of the system. The data at different volumes was thus used to eliminate results contaminated by finite-size effects.

The measured observables were extracted from square $L \times L$ Wilson loops, with $L = 1, 2, \ldots, 9$. Smearing was implemented on the lattice with a parameter $S = L^2/55$ (our convention is that capital letter symbols correspond to lower case symbols in the continuum). For a given Wilson loop, we extracted the coefficients $a_{0,1,2}$ for each orientation and location of the loop. After averaging we obtained one set of coefficients for each gauge configuration. We generated 160 independent gauge configurations at each set of parameters (b, N, V). Consecutive b-values were spaced by increments $\Delta b = 0.001$ and were separated by 1000 passes, half of heat-bath type and half of over-relaxation type. Looking at autocorrelations, we determined that they typically drop by e for every 250 such passes. Averaging over configurations produced coefficients which determined our estimate for $\omega_N(b, L)$, and their errors. The data were sufficiently closely spaced in b that we could use spline interpolation to represent the result as a continuous function of b.

From the HK case we learned that prohibitively large values of N would be needed to display directly the singular large-N behavior. We adopted instead the following strategy: From each $\omega_N(b, L)$ we constructed a number $\tau_N(b, L)$ by solving $\omega_N(b, L) = \omega_N^{\rm HK}(\tau_N(b, L))$. The required inversion is unique. We then showed numerically that the convergence of $\tau_N(b, L)$ to $\tau_\infty(b, L)$ is rapid, like that of the string tension and the deconfinement temperature. Hence, $\omega_\infty(b, L) = \lim_{N\to\infty} \omega_N^{\rm HK}(\tau_N(b, L)) = \lim_{N\to\infty} \omega_N^{\rm HK}(\tau_\infty(b, L))$ and to some universal subleading order in $\frac{1}{N}$, $\omega_N(b, L) \approx \omega_N^{\rm HK}(\tau_\infty(b, L))$. This relation can be taken over to the continuum. Extrapolating to the continuum limit, the two variables b and L get replaced by a single length variable, l, the side of the loop in physical units. We took measurements in a region in which the emerging $\tau_\infty(l)$ extends on both sides of $\tau_\infty = 4$. We numerically determined that $\tau_\infty(l)$ is smooth in l and invertible at $l = l_c$ where $\tau_\infty(l_c) = 4$. We conclude that the continuum $\omega_N(l)$ will develop the same singularity in the vicinity of $l = l_c$ as $\omega_N^{\rm HK}(\tau_\infty(l))$ would, establishing the transition and its universality.

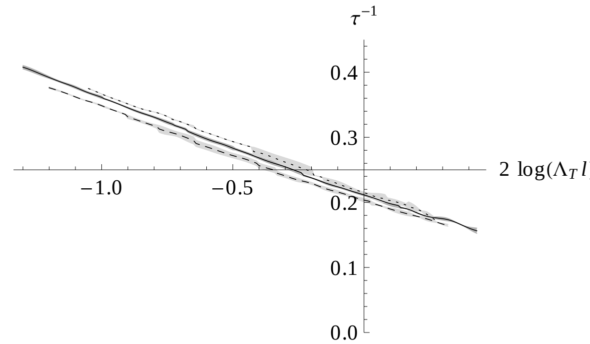

FIG. 3. Rapid convergence to $N = \infty$ for $\tau_N(l)$ in continuum.

The continuum functions $\frac{1}{\tau_N(l)}$ are shown in Fig. 3 for $N = 11, 19, 29$ by a dashed line, a solid line, and a dotted line, respectively, and rapid convergence to $N = \infty$ is evident. The horizontal axis is labeled by $2\log(\Lambda_T l)$. Λ_T is the infinite-N critical deconfinement temperature, roughly equivalent to 264 MeV in QCD units. The shades indicate the accumulated errors, for those regimes where we had enough data to reliably estimate them. By that we mean that we had data points at least at three distinct pairs (b, L) all corresponding to the same physical l. The relation between l and (b, L) was set by a standard one-loop tadpole-improved formula, known to work well from other simulations. A reliable estimate of the continuum limit could be obtained by extrapolating in the lattice spacing squared and observing a linear behavior. At $N = 29$, only two pairs were available for some values of l and a continuum number was obtained by postulating a linear behavior, but without an error. The approximate linearity of the continuum functions $\frac{1}{\tau_N(l)}$ is consistent with asymptotic freedom when $\tau_N(l)$ is viewed as an effective running coupling constant. The slope of the lines in Fig. 3 is about 0.22, while the expected coefficient is about 0.29. The discrepancy has to do with a factor associated with smearing, which has been so far only calculated at tree level where the theory is conformal with a dependence only on the ratio $\frac{l^2}{s}$.

Because of our indirect way to establish criticality, some of the large-N universality predictions we listed

earlier become tautological. However, there remain two extra checks that are meaningful: one is to determine the exponent 3/2 from the ratio a_0/a_1 and the other is the exponent 3/4 from the $N^{-3/4}$ level density in the critical region. Logarithmic fits produce estimates within 1% of the expected values. This concludes our account of the numerical evidence for the non-analyticity.

Having this one example of a large-N phase transition opens up some new questions: Is the transition physical in the particle physics sense, that is, would one actually see new singularities in a large-N, narrow-width approximation to the S-matrix of the theory? If the answer is positive, one might speculate that at $N = \infty$ one does have exactly linear Regge trajectories for high spin states, up to a point, beyond which, presumably, a perturbative behavior ensues. If the answer is negative, one would need to understand how exactly the S-matrix gets shielded from the above large-N phase transition we have established. In this context it is important to note that the non-analyticity we found does not occur in the Wilson loop expectation values themselves, so long as the number of boxes in the Young tableau of the representation is kept finite and fixed as $N \to \infty$.

Many avenues for further investigations are likely to open up, providing some encouragement to those pursuing the long quest of conquering the large-N limit of QCD, albeit, perhaps, in only a semi-analytic way.

We now turn to a review of the history of the subject, focusing on the main contributions. We avoided doing this earlier in order to keep the presentation streamlined. The large-N phase transition we discuss was discovered by Durhuus and Olesen [2] in the context of two-dimensional SU(N) gauge theory, where Casimir dominance is exact. They obtained the phase transition working directly at infinite N, where they identified the inviscid Burgers' equation as playing a central role. Therefore, it would be fair to refer to this transition as the DO transition. Blaizot and Nowak [3] argued that the large-N universality class was controlled by Burgers' equation [4], including its viscous term. This was shown to hold exactly in the HK case [5]. The generic nature of the DO transition, in the sense that it occurs also in three- and four-dimensional SU(N) gauge theory, was conjectured about five years ago [6]. A test in three dimensions was shown to be consistent with the conjecture, but there was no overwhelming numerical evidence [7]. Continuum smearing was introduced at the same time, as a technical device used to define "eigenvalues" of Wilson loops in renormalized QCD. The approximate validity of a "diffusive" viewpoint (that is, using the HK as a model) for the behavior of Wilson loops has been pointed out already in 2005 [8], for the case of the gauge group SU(2). The exact formula for the single eigenvalue-angle density in the HK case at any N was derived in [9]. Casimir dominance has been discussed at the perturbative level in [10] and at the nonperturbative one it was reviewed by Greensite in [11]. The limitation on the size of a finite box at infinite N is studied in [12].

The four-dimensional test in this paper became possible after a hardware upgrade of a computer cluster at Rutgers. Substantial resources were invested to produce a convincing and detailed case in four dimensions, as the question of whether a large-N phase transition can occur in four-dimensional continuum SU(N) gauge theory, as the scale is varied, has been considered repeatedly in the past, but without definitive evidence one way or another [13].

The relation between the large-N expansion and the program of dual topological unitarization [14] was explored a long time ago by Veneziano [15]. The associated question of Regge trajectory linearity has been addressed by McGuigan and Thorn [16]. A more direct connection between the infinite-N limit and zero string-coupling string theory has been an object of study for a long time, and perhaps the best known line of attack is the suggestion to start from the Makeenko-Migdal [17] equations and guess a solution defined by a string theory.

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